Successive Differentiation

B.Sc (Hons) Part I

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Study Material :→

Notation :→ $y = f(x)$

$y_1 = \frac{dy}{dx} = f'(x) = y'$

: Differential co-efficient of y with respect to x

= 1st order Derivative

$y_2 = \frac{d^2y}{dx^2}$ = 2nd order Derivative of $y = F(x)$

$y_n = \frac{d^ny}{dx^n}$ = nth order Derivative of $y = F(x)$

nth Differential Co-efficient of Some Standard Functions —

I. nth Derivative of x^m

Let $y = x^m$

then $y_1 = m x^{m-1}$

$y_2 = m(m-1)x^{m-2}$

$y_3 = m(m-1)(m-2)x^{m-3}$

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$y_n = m(m-1)(m-2)\cdots(m-n+1)x^{m-n}$

thus, if $y = x^m$

$\Rightarrow y_n = m(m-1)(m-2)\cdots(m-n+1)x^{m-n}$

Find the nth Differential co-efficient of

$$y = (ax+b)^n$$

Let $y = (ax+b)^n$

$\Rightarrow y_1 = a\,n(ax+b)^{n-1}$

$y_2 = a^2 n(n-1)(ax+b)^{n-2}$

$y_3 = a^3 n(n-1)(n-2)(ax+b)^{n-3}$

$\cdots$

$y_n = a^n n(n-1)(n-2)\cdots 2\cdot 1\,(ax+b)^{n-n}$

$= a^n n(n-1)(n-2)\cdots 2\cdot 1$

$= a^n \lfloor n$

Find the nth Derivative of

$$y = \frac{1}{ax+b}$$

Let $y = \frac{1}{ax+b} = (ax+b)^{-1}$

$\Rightarrow y_1 = (-1)\,a\,(ax+b)^{-2}$

$y_2 = (-1)^2\, 1\cdot 2\cdot a^2\,(ax+b)^{-3}$

$y_3 = (-1)^3\, 1\cdot 2\cdot 3\cdot a^3\,(ax+b)^{-4}$

$\cdots$

$y_n = (-1)^n\, 1\cdot 2\cdot 3\cdots n\; a^n\,(ax+b)^{-(n+1)}$

$= \frac{(-1)^n \lfloor n\; a^n}{(ax+b)^{n+1}}$

Find the n th differential co-efficient of $\log(ax+b)$

Sol^n Here $y = \log(ax+b)$

$$\Rightarrow y_1 = \frac{1}{ax+b}(a)$$

$$\Rightarrow y_1 = a \cdot (ax+b)^{-1}$$

$$y_2 = a^2(-1)(ax+b)^{-2}$$

$$y_3 = a^3(-1)^2 1\cdot 2(ax+b)^{-3}$$

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$$y_n = \frac{a^n (-1)^{n-1} 1\cdot 2\cdot 3 \cdots (n-1)}{(ax+b)^n}$$

$$= \frac{(-1)^{n-1} \lfloor n-1}{(ax+b)^n} \text{ Ans.}$$

Similarly if $y = \sin(ax+b)$

$$\Rightarrow y_n = a^n \sin\left(ax+b+n\frac{\pi}{2}\right)$$

Since $y = \sin(ax+b)$

$$\Rightarrow y_1 = a \cdot \cos(ax+b)$$

$$= a \sin\left[\pi/2 + ax+b\right]$$

$$y_2 = a^2 \sin\left[2\frac{\pi}{2} + ax+b\right]$$

$$y_n = a^n \sin(ax+b+n\pi/2)$$

Similary if $y = \cos(ax+b)$

then $y_n = a^n \cos\left[ax+b+\frac{n\pi}{2}\right]$

Find the nth Derivative of

$e^{ax} \sin(bx+c)$

Here $y = e^{ax} \sin(bx+c)$

$\Rightarrow y_1 = a e^{ax} \sin(bx+c) + e^{ax} \cdot b\cos(bx+c)$

$= e^{ax} \left[a \sin(bx+c) + b\cos(bx+c)\right]$

Let $a = r\cos\theta$ $\quad b = r\sin\theta$

$\therefore a^2+b^2 = r^2 \Rightarrow r = \sqrt{a^2+b^2}$

$= e^{ax}\sqrt{a^2+b^2}\left[\frac{a \sin(bx+c)}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}} \cos(bx+c)\right]$

$= e^{ax}\sqrt{a^2+b^2}\left[\sin(bx+c+\theta)\right]$

$= e^{ax}(a^2+b^2)^{\frac{1}{2}} \sin(bx+c+\theta)$ Where

$\theta = \tan^{-1}\frac{b}{a}$

$y_2 = e^{ax} r^2 \sin(bx+c+2\theta)$

$\therefore y_n = e^{ax} r^n \sin(bx+c+n\theta)$ which is the required Result

Where $\theta = \tan^{-1}\frac{b}{a}$

$r = \sqrt{a^2+b^2} = (a^2+b^2)^{\frac{1}{2}}$

$\Rightarrow r^n = (\sqrt{a^2+b^2})^n = a^2+$

Problem based on Successive Differentiation

Problem 1. If $y = e^{a\sin^{-1}x}$ then Prove that-

$$(1-x^2)y_2 - xy_1 = a^2y$$

Solution :-> $y = e^{a\sin^{-1}x}$

$$\Rightarrow y_1 = e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}}$$

$$\Rightarrow y_1 = \frac{ay}{\sqrt{1-x^2}} \quad \text{put } y = e^{a\sin^{-1}x}$$

$$\Rightarrow y_1^2(1-x^2) = a^2y^2 \quad \text{[squaring on both sides]}$$

Diff. again with respect to x

$$2y_1y_2(1-x^2) - 2xy_1^2 = 2a^2y \cdot y_1$$

$$\left[\text{using } y = f(x)\cdot g(x) \Rightarrow \frac{dy}{dx} = g(x)\frac{d}{dx}f(x) + f(x)\cdot\frac{d}{dx}g(x)\right]$$

$$\Rightarrow (1-x^2)y_2 - xy_1 = a^2y.$$

Proved

Problem 2. If $y = \sin(m\sin^{-1}x)$ Prove that-

$$(1-x^2)y_2 - xy_1 + m^2y = 0$$

Soln

Here By The question

$$y = \sin(m\sin^{-1}x)$$

Diff. w.r.t. respect to x

$$\frac{dy}{dx} = \frac{d\sin(m\sin^{-1}x)}{d(m\sin^{-1}x)} \cdot m\frac{d\sin^{-1}x}{dx}$$

$$y_1 = \cos(m\sin^{-1}x) \cdot \frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow y_1^2 = \frac{m^2\cos^2(m\sin^{-1}x)}{(1-x^2)}$$ [squaring on both sides]

$$= \frac{m^2[1-\sin^2(m\sin^{-1}x)]}{1-x^2}$$

$$\Rightarrow y_1^2(1-x^2) = m^2(1-y^2)$$ put $y = \sin(m\sin^{-1}x)$

Differentiating w.r.t. respect to x again.

$$\cancel{2y_1}y_2(1-x^2) - \cancel{2}xy_1^2 = m^2(-\cancel{2}y\cancel{y_1})$$

$$\Rightarrow (1-x^2)y_2 - xy_1 = -m^2y$$

$$\Rightarrow (1-x^2)y_2 - xy_1 + m^2y = 0$$

Which is the required result.

If : $y = \frac{1}{x^2+a^2}$ then Find y_n

Sol^n :→ Here $y = \frac{1}{x^2+a^2}$

$$= \frac{1}{(x+ia)(x-ia)}$$

Now Resolve in to partial Fraction

$$\frac{1}{(x+ia)(x-ia)} = \frac{1}{2ia}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$$

using $y = \frac{1}{ax+b}$

Then $y_n = \frac{(-1)^n a^n \lfloor n}{(ax+b)^{n+1}}$

$$= \frac{1}{2ia}\left[\frac{(-1)^n \lfloor n}{(x-ia)^{n+1}} - \frac{(-1)^n \lfloor n}{(x+ia)^{n+1}}\right]$$

$$= \frac{1}{2ia}(-1)^n \lfloor n \left[\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right]$$

Now to semplify this

We use DeMoiver's Theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

put $x = r\cos\theta$ $\quad \tan\theta = \frac{a}{x}$

$a = r\sin\theta$ $\quad \Rightarrow \theta = \tan^{-1}\frac{a}{x}$

$r^2 = x^2 + a^2$

$$\therefore y_n = \frac{(-1)^n \lfloor n}{2ia}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^{n+1}} - \frac{1}{(r\cos\theta + ir\sin\theta)^{n+1}}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia}\left[\frac{1}{r^{n+1}[\cos(n+1)\theta - i\sin(n+1)\theta]} - \frac{1}{r^{n+1}[\cos(n+1)\theta + i\sin(n+1)\theta]}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia} \cdot \frac{1}{r^{n+1}}\left[[\cos(n+1)\theta + i\sin(n+1)\theta] - [\cos(n+1)\theta - i\sin(n+1)\theta]\right]$$

$$= \frac{(-1)^n \lfloor n}{\cancel{2i}a} \cdot \frac{1}{r^{n+1}}\left[\cancel{2i}\sin(n+1)\theta\right]$$

$$= \frac{(-1)^n \lfloor n \sin(n+1)\theta}{a \cdot r^{n+1}}$$

$$= \frac{(-1)^n \lfloor n \sin(n+1)\theta}{ar \cdot r^n}$$

$$\because a = r\sin\theta$$

$$\Rightarrow r = \frac{a}{\sin\theta}$$

$$= \frac{(-1)^n \lfloor n \sin(n+1)\theta}{a \cdot (a\ldots)^{n+1}}$$

$\therefore y_n = \dfrac{(-1)^n \lfloor n \sin(n+1)\theta}{a^{n+2}} \sin^{n+1}\theta.$

thus if $y = \dfrac{1}{x^2+a^2}$

$\Rightarrow y_n = \dfrac{(-1)^n \lfloor n \sin(n+1)\theta}{a^{n+2}} \cdot \sin^{n+1}\theta$